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# FULLY NONLINEAR GÖRTLER VORTICES IN CONSTRICTED CHANNEL FLOWS AND THEIR EFFECT ON THE ONSET OF SEPARATION

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## ABSTRACT

The development of fully nonlinear Görtler vortices in the high Reynolds number flow in a symmetrically constricted channel is investigated. Attention is restricted to the case of ‘strongly’ constricted channels considered by Smith and Daniels (1981) for which the scaled constriction height is asymptotically large. Such flows are known to develop a Goldstein singularity and subsequently become separated at some downstream station past the point of maximum channel constriction. It is shown that these flows can support fully nonlinear Görtler vortices, of the form elucidated by Hall and Lakin (1988), for constrictions which have an appreciable region of local concave curvature upstream of the position at which separation occurs. The effect on the onset of separation due to the nonlinear Görtler modes will be discussed. A brief discussion of other possible nonlinear states which may also have a dramatic effect in delaying (or promoting) separation will be given.

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## 1. Introduction

Our concern is with the development of fully nonlinear Görtler vortices in Smith–Daniels flows and their effect on the onset of separation. By Smith–Daniels flows we refer to the steady flow in a symmetrically constricted channel considered by Smith and Daniels (1981) who showed that when the scaled height,  $h \gg 1$ , of the constriction becomes large a secondary classical boundary layer of thickness  $O(h^{-1/2})$  is set up within the wall boundary layer (of thickness  $O(Re^{-1/3})$  where  $Re$  is the global Reynolds number for the flow). This inner, classical boundary layer develops a Goldstein (1948) singularity at a position downstream of the position of maximum channel constriction. However, Smith and Daniels demonstrated that this singularity could be removed, without any upstream influence being induced, and their analysis was able to describe the reattachment of the boundary layer at a distance  $O(h^3)$  downstream.

Subsequently Hall and Bennett (1986) considered the linear instability of this flow, in the general context of the stability of both steady and unsteady interactive boundary layer flows, at stream-wise locations before the Goldstein singularity develops. This study demonstrated that the flow is unstable to Görtler vortices for channels in which the constriction has a region of concave curvature, in an interval before the position at which separation occurs. In this analysis  $f_{xx}$ , where  $f(x) = hF(x)$  with  $h \gg 1$  is the constriction profile, effectively plays the role of the Görtler number.

In the case of external boundary layer flows over a wall of variable curvature Hall and Lakin (1988), motivated by the earlier weakly nonlinear theory of Görtler vortices by Hall (1982b) and the numerical solution of the fully nonlinear Görtler vortex governing equations by Hall (1988), developed a self consistent asymptotic theory for short wavelength, large amplitude Görtler vortices. In this theory the Görtler number  $G$  scales with  $k^4$  (where  $k \gg 1$  is the vortex wave number). Their analysis demonstrated that the mean flow adjusts due to the presence of the vortex so as to render all vortex modes neutrally stable. Within an  $O(1)$  region of the boundary layer the mean flow is driven by the vortex velocity field and as such has no relation to the unperturbed boundary layer flow. Such a situation is similar to that postulated by Malkus (1956) for turbulent flows; Malkus argued

that the mean part of a turbulent flow would adjust so as to ensure that all ‘modes’ are rendered neutrally stable.

As pointed out by Hall and Lakin (1988), the modification of the mean flow by the nonlinear longitudinal vortex motion raises the question as to what effect this modified mean flow has on the onset of separation, in separating flows that can support Görtler vortex like disturbances. One important result of the Hall and Lakin calculation is that the skin friction for the vortex induced mean flow can be dramatically increased over that of the unperturbed boundary layer (although no quantitative results on this point are presented this can be inferred from their figure 6, Hall and Lakin (1988) pp 440–441); we would then expect that the onset of separation could be considerably delayed due to the presence of the vortex motion. In light of the work of Hall and Bennett (1986) alluded to above, we choose to concentrate on the Smith and Daniels (1981) problem of the steady flow in a symmetrically constricted channel.

The procedure adopted for the rest of this paper is as follows. In §2 we develop the governing equations for nonlinear Görtler vortices in the  $O(h^{-1/2})$  classical boundary layer of Smith and Daniels together with a discussion of the relevant results from Smith (1982) concerning the onset of separation as well as the those points from the linear theory of Hall and Bennett (1986) pertinent to the present investigation. In §3 we extend the linear theory of Hall and Bennett (1986) to the fully nonlinear regime for short wavelength vortices considered by Hall and Lakin (1988). In §4 we present the numerical solution of the vortex induced mean flow equations. Finally in §5 a discussion of our results is presented; particular emphasis is given to the effect that the presence of the nonlinear vortex structure has upon the onset of separation.

## 2. Formulation of the governing equations

Consider the steady motion of an incompressible fluid of density  $\rho$  and kinematic viscosity  $\nu$  in a symmetrically constricted channel. Let  $L$  denote the typical length scale of the constriction and let  $U_\infty$  denote the typical free-stream speed sufficiently far upstream of the constriction (see figure 1). We define the Reynolds number  $Re$  by  $Re = U_\infty L/\nu$ , and throughout this work we will assume that the flow is such that the Reynolds number

is asymptotically large. The steady three-dimensional Navier-Stokes equations in nondimensional form are

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} + Re^{-1} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \\
u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= -\frac{\partial p}{\partial y} + Re^{-1} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \\
u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + Re^{-1} \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right),
\end{aligned} \tag{2.1}$$

where  $(x, y, z)$  are the usual Cartesian co-ordinates, nondimensionalized with respect to  $L$ ,  $(u, v, w)$  are the corresponding velocity components, nondimensionalized with respect to the upstream velocity  $U_\infty$  and  $p$  is the pressure, nondimensionalized with respect to  $\rho U_\infty^2$ . For definiteness we will assume, following Smith (1982), sufficiently far upstream of the channel constriction the flow is Poiseuille

$$u \rightarrow y - y^2, \quad v \rightarrow 0, \quad p \rightarrow -\frac{2x}{Re}, \quad x \rightarrow -\infty.$$

As shown by Smith (1982), with a constriction length  $L = O(1)$ , the critical height of the constriction is  $O(Re^{-1/3})$  (Smith's 'strong constriction'). We then take the constriction  $y = hRe^{-1/3}F(x)$  at the lower wall (a corresponding constriction  $y = 1 - hRe^{-1/3}F(x)$  is also present at the upper wall, however here we are predominantly concerned with the boundary layer flow in the vicinity of the lower wall and as such we will restrict our attention to the lower boundary); we will subsequently take the limit  $h \gg 1$  (this being the appropriate asymptotic regime of the Smith and Daniels (1981) investigation) however we initially consider  $0 < h < \infty$ . The flow develops a viscous wall layer of thickness  $O(Re^{-1/3})$ , in which the velocity field and pressure expand as (Smith (1982))

$$\begin{aligned}
(u, v, w, p) &= (Re^{-1/3}U, Re^{-2/3}V, Re^{-2/3}W, Re^{-2/3}P_0 + Re^{-4/3}P_1), \\
x &= O(1), \quad y = Re^{-1/3}Y, \quad z = Re^{-1/3}Z,
\end{aligned} \tag{2.2}$$

where  $P_0$  is a function of  $x$  while  $P_1$  is a function of  $x, Y, Z$ . The governing equations in the viscous wall layer are then given by

$$\begin{aligned}\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} &= 0, \\ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial Y} + W \frac{\partial U}{\partial Z} &= -\frac{\partial P_0}{\partial x} + \frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2}, \\ U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial Y} + W \frac{\partial V}{\partial Z} &= -\frac{\partial P_1}{\partial Y} + \frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2}, \\ U \frac{\partial W}{\partial x} + V \frac{\partial W}{\partial Y} + W \frac{\partial W}{\partial Z} &= -\frac{\partial P_1}{\partial Z} + \frac{\partial^2 W}{\partial Y^2} + \frac{\partial^2 W}{\partial Z^2},\end{aligned}\tag{2.3}$$

together with the boundary conditions

$$\begin{aligned}(u, v, w, p) &\rightarrow (Y, 0, 0, 0) \quad \text{as} \quad x \rightarrow -\infty, \\ u = v = w &= 0 \quad \text{on} \quad Y = hF(x), \\ U &\sim Y \quad Y \rightarrow \infty.\end{aligned}\tag{2.4}$$

Here we have anticipated the well known result that the Görtler vortex wavelength is scaled on the boundary layer thickness.

In the asymptotic limit  $h \gg 1$ , Smith and Daniels (1981) have demonstrated that the flow described above develops a classical boundary layer of thickness  $O(h^{-1/2})$  attached to the hump. In the absence of any vortex component the appropriate scalings are found to be

$$U = h\bar{u}, \quad V = h^{1/2}\bar{v}, \quad P_0 = h^2\bar{p} \quad \xi = h^{1/2}Y.$$

From the analysis of Smith and Daniels (1981) the governing equations are given by

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial \xi} = F F_x + \frac{\partial^2 \bar{u}}{\partial \xi^2}, \quad \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial \xi} = 0,\tag{2.5a, b}$$

together with the boundary conditions

$$\bar{u} = \bar{v} = 0, \quad \xi = 0, \quad \bar{u} \rightarrow F \quad \xi \rightarrow \infty,\tag{2.6a, b}$$

and the upstream conditions

$$\bar{u}, \bar{v} \rightarrow 0 \quad x \rightarrow -\infty;\tag{2.6c}$$



(here we are assuming  $F(x) \rightarrow 0$  as  $x \rightarrow -\infty$  and we have made use of the Prandtl transformation). We note that in (2.5a) the pressure gradient term  $-FF_x$  arises from the outer flow solutions (the reader is referred to Smith and Daniels (1981) for full details). Subsequently we will assume that the constriction profile is such that  $F(x) = 0$  for  $x \leq 0$  so that the upstream condition (2.6c) can be replaced by

$$\bar{u}, \bar{v} = 0 \quad x \leq 0. \quad (2.6d)$$

To derive the governing equations for the Görtler vortex velocity field, which we assume is confined to the secondary  $O(h^{-1/2})$  boundary layer, the following scalings are employed (see Hall and Bennett (1986)). The  $x, y$  and  $z$  disturbance velocity components scale with the  $x, y$  and  $y$  basic state velocity components; the  $y$  and  $z$  variations are on the classical boundary layer scale, namely  $O(h^{-1/2})$ , and the pressure perturbations are reduced in size until the pressure gradient and viscous terms in the  $y$  and  $z$  momentum equations become comparable. Thus

$$\begin{aligned} \xi &= h^{1/2}Y, & z &= h^{1/2}Z, \\ U &= hu, & V &= h^{1/2}v, & W &= h^{1/2}w, & P_0 &= h^2\bar{p}, & P_1 &= hp. \end{aligned} \quad (2.7)$$

The governing equations are then given by

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial \xi} + \frac{\partial w}{\partial z} &= 0, \\ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial \xi} + w \frac{\partial u}{\partial z} &= FF_x + \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial z^2}, \\ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial \xi} + w \frac{\partial v}{\partial z} + h^{3/2}F_{xx}u^2 &= -\frac{\partial p}{\partial \xi} + \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial z^2}, \\ u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial \xi} + w \frac{\partial w}{\partial z} &= -\frac{\partial p}{\partial z} + \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial z^2}, \end{aligned} \quad (2.8a-d)$$

together with the boundary conditions

$$\begin{aligned} u &= v = w = 0 & \xi &= 0, \\ u &\rightarrow F, & v_\xi, w &\rightarrow 0 & \xi &\rightarrow \infty, \\ u, v, w &= 0 & x &\leq 0, \end{aligned} \quad (2.9a-c)$$

where we have made use of the Prandtl transformation.

Our concern is with the solution of the nonlinear system (2.8,9) in the limit of short spanwise wavelength,  $\partial/\partial z \gg 1$ . Before proceeding with a description of the solution of these equations we make some comments concerning the unperturbed system (2.5,6) and recall the relevant details from the linear theory of Hall and Bennett (1986).

In practice the solution of the system (2.5,6) is a numerical task; we briefly describe the numerical technique used to solve this system. In order to obtain solutions of (2.5a,c) to within the numerical accuracy required for the subsequent calculations of §4 it was found to be advantageous to solve a modified system obtained from (2.5) by writing

$$\bar{u} = F\bar{u}_0, \quad \bar{v} = -F_x\xi + \bar{v}_0,$$

where  $(\bar{u}_0, \bar{v}_0)$  are governed by

$$\begin{aligned} F_x\bar{u}_0^2 + F\bar{u}_0\frac{\partial\bar{u}_0}{\partial x} + (\bar{v}_0 - F_x\xi)\frac{\partial\bar{u}_0}{\partial\xi} &= F_x + \frac{\partial^2\bar{u}_0}{\partial\xi^2}, \\ F_x(\bar{u}_0 - 1) + F\frac{\partial\bar{u}_0}{\partial x} + \frac{\partial\bar{v}_0}{\partial\xi} &= 0, \end{aligned}$$

with the appropriate free-stream boundary condition now given by

$$\bar{u}_0 \rightarrow 1 \quad \xi \rightarrow \infty;$$

all other boundary conditions remaining unchanged. This modified system was discretized according to

$$\begin{aligned} \frac{Fu_n\tilde{u}_n}{\epsilon} - \frac{1}{h^2}(\tilde{u}_{n+1} - 2\tilde{u}_n + \tilde{u}_{n-1}) &= \frac{Fu_n^2}{\epsilon} + F_x(1 - u_n^2) \\ &\quad - \frac{(v_n - F_x\xi_n)(u_{n+1} - u_{n-1})}{2h}, \quad (2.10a, b) \\ \frac{\tilde{v}_{n+1} - \tilde{v}_{n-1}}{2h} &= \frac{F(u_n - \tilde{u}_n)}{\epsilon} - F_x(\tilde{u}_n - 1), \end{aligned}$$

where a tilde denotes a quantity evaluated at  $x + \epsilon$  and the index  $n$  refers to a quantity evaluated at the grid point  $\xi = nh$ ; we explicitly impose the boundary conditions (2.6a,b,d). Given  $u_n, v_n$  and  $F$  the tridiagonal system (2.10a) can be solved for the updated value  $\tilde{u}_n$ ;  $\tilde{v}_n$  is then determined from (2.10b). The solution was marched downstream from  $x = 0$ ,

with the initial conditions (2.6d), iterating at each step until converged values of  $\tilde{u}$  and  $\tilde{v}$  were obtained. For definiteness we will restrict our attention to the hump profile

$$F(x) = \begin{cases} 0, & x \leq 0, \\ \alpha e^{-1}(x^4/16 - x^3 + 4x^2), & 0 \leq x \leq 4, \\ \alpha x^2 e^{-x^2/16}, & x \geq 4, \end{cases} \quad (2.11)$$

where  $\alpha$  is an amplitude parameter. Consider now the boundary conditions (2.9b). As noted by Smith (1982) the boundary layer is accelerated where  $F_x(x) > 0$  (upstream of the point of maximum constriction  $x = x_{\max} = 4$ ). Downstream of  $x_{\max}$  the flow is decelerated due to the adverse pressure gradient  $-FF_x$ ; the skin friction then tends to zero, the Goldstein singularity is encountered and the boundary layer flow becomes separated. In figure 2 we present a plot of the skin friction  $\tau(x) = \bar{u}_\xi(x, 0)$  versus  $x$  for the profile (2.11) and the particular choice  $\alpha = 1.031$ . We see that  $\tau$  attains a maximum at  $x \approx 2.9$  and then decreases until  $\tau = 0$  at  $x = x_s \approx 4.53$  at which point the Goldstein singularity will arise and separation will occur. We note that changing the amplitude  $\alpha$  has no effect on either the position at which  $\tau$  attains its maximum nor the point at which  $\tau = 0$ . Our interest lies in the effect that the presence of a fully nonlinear vortex structure upstream of  $x_{\max}$  may have in delaying (or promoting) the onset of separation.

The linear theory of Hall and Bennett (1986) demonstrates that the boundary layer becomes unstable to linear Görtler vortices at the position  $x^*$  where

$$2F_{xx}\bar{u}\frac{\partial\bar{u}}{\partial\xi}\bigg|_{(x^*,\xi^*)} = 1, \quad (2.12)$$

and are confined to a thin layer about the position  $\xi = \xi^*$  where

$$\frac{\partial}{\partial\xi}\left(\bar{u}\frac{\partial\bar{u}}{\partial\xi}\right)\bigg|_{\xi=\xi^*} = 0. \quad (2.13)$$

As noted by Hall and Bennett, for humps with  $F > 0$  (ie channel constriction)  $\bar{u}\bar{u}_\xi$  remains positive until the Goldstein singularity develops, at  $x = x_s$  say, and hence for instability we require the constriction to be locally concave somewhere in the interval  $0 \leq x < x_s$ . The profile (2.11) was chosen with this in mind; it is locally concave in the region  $0 \leq x \leq 1.69$ .

As an aside, we note that, from the results of Hall and Bennett, there is a critical amplitude  $\alpha$  below which instability *cannot* occur (see expression (3.15) and the preceding discussion in Hall and Bennett (1986)). In figure 3 we present a plot of the neutral position  $x^*$  versus the amplitude  $\alpha$  for the profile (2.11); we find the critical amplitude to be  $\alpha = \alpha_c \approx 0.84$ . For  $\alpha > \alpha_c$  there is a finite interval  $x_0^* \leq x^* \leq x_1^*$  in which the flow can support linearly unstable Görtler modes and this interval increases with increasing  $\alpha$ .

### 3. Evolution equations for large amplitude Görtler vortices

We now proceed to develop an asymptotic solution of the governing equations (2.8a-d) valid in the short wavelength limit. We assume that the solution  $(\bar{u}(x, \xi), \bar{v}(x, \xi))$  of the classical boundary layer equations (2.5a,b) becomes linearly unstable to Görtler vortices at  $x = x^*$  where  $x^*$  is to be determined from (2.12).

In the  $O(h^{-1/2})$  boundary layer the effective Görtler number  $G$  becomes  $O(h^{3/2})$  so that, on the basis of the nonlinear theory of Hall and Lakin (1988), we anticipate vortices with wavenumber  $k = O(h^{3/8})$  (the scaling  $G = O(k^4)$  is that which is appropriate to the right-hand-branch of the linear neutral curve, see Hall (1982)). At first sight these scalings appear to be in disagreement with those presented in Hall and Bennett (1986); in fact, there is no discrepancy when one considers the boundary layer scalings (2.7). The analysis of Hall and Lakin demonstrates that in the presence of the nonlinear vortex state the flow regime breaks up into three regions; a central core region, in which the mean flow is induced by the vortex velocity field, which is bounded above and below by thin shear layers in which the vortex amplitude is reduced to zero. Outside the region of vortex activity the mean flow is governed by the unperturbed boundary layer equations (see figure 4). We now proceed to develop the corresponding structure for the flow described in §2; we emphasize that the nonlinear vortex state will be confined wholly within the  $O(h^{-1/2})$  boundary layer. Hence, we are anticipating that the only effect on the onset of separation due to the presence of the nonlinear vortex state is that the *position* at which separation occurs will be changed; the asymptotic analysis of Smith and Daniels concerning the removal of the Goldstein singularity still remains valid. As the analysis mirrors that of Hall and Lakin we

will give only the salient features of that calculation (the reader is referred to that paper for full details).

To proceed we first consider the core region, in which the vortex amplitude is chosen so as to have an  $O(1)$  effect on the mean flow in this region. The appropriate expansions in the core region are then given by

$$\begin{aligned} u &= \bar{u}_0 + h^{-3/8} \left\{ U_{01} \cos h^{3/8} z + \dots \right\}, \\ v &= \bar{v}_0 + h^{3/8} \left\{ V_{01} \cos h^{3/8} z + \dots \right\}, \end{aligned} \quad (3.1)$$

together with similar expansions for  $w$  and  $p$ ; here  $\dots$  denotes terms which do not enter into the subsequent analysis. Substituting the expansions (3.1) into the governing equations gives, for the vortex components

$$U_{01} + \bar{u}_{0\xi} V_{01} = 0, \quad V_{01} + 2\bar{u}_0 F_{xx} U_{01} = 0, \quad (3.2a, b)$$

which for consistency requires

$$2F_{xx} \bar{u}_0 \bar{u}_{0\xi} = 1. \quad (3.3)$$

Integrating (3.3) yields, for the vortex induced mean flow

$$\bar{u}_0 = \sqrt{\frac{A(x) + \xi}{F_{xx}}}, \quad (3.4a)$$

while from the continuity equation we have

$$\bar{v}_0 = -B(x) - A_x \sqrt{\frac{A(x) + \xi}{F_{xx}}} + \frac{F_{xxx}}{3} \left( \frac{A(x) + \xi}{F_{xx}} \right)^{3/2}; \quad (3.4b)$$

here  $A(x)$  and  $B(x)$  are to be determined. Thus, in the core region the mean flow is found from a solvability condition on the vortex amplitude equations. At this stage the vortex amplitude  $V_{01}$  remains undetermined; this is found from the mean flow component of the first momentum equation. Thus

$$\bar{u}_0 \frac{\partial \bar{u}_0}{\partial x} + \bar{v}_0 \frac{\partial \bar{u}_0}{\partial \xi} - \frac{\partial^2 \bar{u}_0}{\partial \xi^2} - F F_x = \frac{1}{2} \frac{\partial}{\partial \xi} \left( \frac{\partial \bar{u}_0}{\partial \xi} V_{01}^2 \right),$$

which making use of (3.4) can be integrated to give

$$C(x) + \frac{1}{4\sqrt{F_{xx}(A+\xi)}} V_{01}^2 = -B\sqrt{\frac{A+\xi}{F_{xx}}} - \frac{F_{xxx}(A+\xi)^2}{6F_{xx}^2} - \frac{1}{2\sqrt{F_{xx}(A+\xi)}} - FF_x\xi. \quad (3.5)$$

If we let  $\xi_j(x)$  ( $j = 1, 2$ ) denote the outer limits of the region of vortex activity we then have from (3.5), upon eliminating  $C(x)$ , a relation between  $\xi_1$  and  $\xi_2$ , namely

$$\begin{aligned} -B\sqrt{\frac{A+\xi_1}{F_{xx}}} - \frac{F_{xxx}(A+\xi_1)^2}{6F_{xx}^2} - \frac{1}{2\sqrt{F_{xx}(A+\xi_1)}} - FF_x\xi_1 = & -B\sqrt{\frac{A+\xi_2}{F_{xx}}} \\ & - \frac{F_{xxx}(A+\xi_2)^2}{6F_{xx}^2} - \frac{1}{2\sqrt{F_{xx}(A+\xi_2)}} - FF_x\xi_2. \end{aligned} \quad (3.6)$$

Thus, the vortex amplitude in the core region is reduced to zero as  $\xi \rightarrow \xi_{1,2}$ ; however we require thin shear layers centred on  $\xi_{1,2}$  in order to smooth out the algebraically decaying vortex component. We note that at this point of the analysis we are unable to determine the functions  $A(x)$ ,  $B(x)$ ,  $\xi_1(x)$  and  $\xi_2(x)$ .

We now consider the solution in the shear layers. As noted by Hall and Lakin, the thickness of these layers is found by balancing diffusion across the layer with convection in the stream-wise direction. In the case under consideration such a balance shows the thickness of these layers to be  $O(h^{-1/4})$ . We then write

$$\eta = h^{1/4}(\xi - \xi_2),$$

and in the shear layer we replace  $\partial/\partial x$  and  $\partial/\partial \xi$  by  $\partial/\partial x - h^{1/4}\xi_2'\partial/\partial \eta$  and  $h^{1/4}\partial/\partial \eta$ , respectively. We will subsequently restrict our attention to the shear layer centred on  $\xi_2$ ; an identical analysis holds in the layer centered on  $\xi = \xi_1$ .

From the core solutions (3.4-5) the appropriate expansions in this shear layer are found to be

$$\begin{aligned} u &= \bar{u}_0 + h^{-1/4}\bar{u}_1 + h^{-1/2}\bar{u}_2 + \dots + h^{-1/2} \left\{ (u_{01} + h^{-1/4}u_{11} + \dots) \cos h^{3/8}z + \dots \right\}, \\ v &= \bar{v}_0 + h^{-1/4}\bar{v}_1 + \dots + h^{1/4} \left\{ (v_{01} + h^{-1/4}v_{11} + \dots) \cos h^{3/8}z + \dots \right\}, \end{aligned}$$

together with similar expansions for  $w$  and  $p$ . Substitution of these expansions into the governing equations we find that, after some trivial manipulations, the leading order mean flow terms are given by

$$\begin{aligned}\bar{u}_0 &= \sqrt{\frac{A + \xi_2}{F_{xx}}}, \quad \bar{u}_1 = \frac{\eta}{2\sqrt{F_{xx}(A + \xi_2)}}, \\ \bar{v}_0 &= -B(x) - A_x \sqrt{\frac{A + \xi_2}{F_{xx}}} + \frac{F_{xxx}}{3} \left( \frac{A + \xi_2}{F_{xx}} \right)^{3/2},\end{aligned}\tag{3.7a, c}$$

where we have made use of matching with the core solutions in the limit  $\eta \rightarrow -\infty$ . The governing equations for the leading order vortex amplitudes  $(u_{01}, v_{01})$  are found to be consistent whereas the equations for  $(u_{11}, v_{11})$  are consistent provided that

$$\frac{\partial^2 v_{01}}{\partial \eta^2} - \frac{S(x)\eta v_{01}}{3} = \frac{1}{3}v_{01}^3 + f_1(x)v_{01},$$

which gives the governing equation for the vortex amplitude  $v_{01}$ . Here

$$S = -2B - \frac{4F_{xxx}}{3} \left\{ \frac{A + \xi_2}{F_{xx}} \right\}^{3/2} + \frac{1}{A + \xi_2} - 4F_x \sqrt{F_{xx}(A + \xi_2)},$$

and  $f_1$  is a function of  $x$ , the precise nature of which is not required here. This equation is a modified form of the second Painlevé transcendent and has been shown to have a unique solution, Hastings and McLeod (1980), with the asymptotic behaviour

$$v_{01}^2 \sim S\eta, \quad \eta \rightarrow -\infty, \quad v_{01} \rightarrow 0, \quad \eta \rightarrow \infty.$$

It then follows that the vortex is constrained to lie below  $\xi = \xi_2$  (a similar analysis of the shear layer about  $\xi_1$  shows that the vortex is constrained to lie above  $\xi = \xi_1$ ) so that the nonlinear vortex is confined to the  $O(1)$  core region. We note from (3.7a–c) that the mean flow in the shear layer is essentially unaltered by the presence of the vortex (in fact, the first two terms in the stream-wise mean flow expansion in this layer are obtained from the Taylor series expansion of the core flow about  $\xi_2$ ). Hence the mean flow outside the core region must have  $\bar{u}$ ,  $\bar{u}_\xi$  and  $\bar{v}$ , in the limits  $\xi \rightarrow \xi_1^-, \xi_2^+$ , defined by the core solution evaluated at  $\xi = \xi_{1,2}$  respectively.

Outside the region of vortex activity there is only a *mean* velocity field and we write

$$u = \bar{u} + O(h^{-1/2}), \quad v = \bar{v} + O(h^{-1/2}),$$

where  $(\bar{u}, \bar{v})$ , defined in  $(0, \xi_1)$  and  $(\xi_2, \infty)$ , satisfy

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial \xi} = \frac{\partial^2 \bar{u}}{\partial \xi^2} + F F_x, \quad \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial \xi} = 0. \quad (3.8)$$

This system is to be solved subject to

$$\bar{u} = \bar{v} = 0, \quad \xi = 0, \quad \bar{u} \rightarrow F, \quad \xi \rightarrow \infty, \quad (3.9a, b)$$

and

$$\begin{aligned} \bar{u}(\xi_j) &= \sqrt{\frac{A + \xi_j}{F_{xx}}}, \quad \bar{u}_\xi(\xi_j) = \frac{1}{2\sqrt{F_{xx}(A + \xi_j)}}, \\ \bar{v}(\xi_j) &= -B(x) - A_x \sqrt{\frac{A + \xi_j}{F_{xx}}} + \frac{F_{xxx}}{3} \left( \frac{A + \xi_j}{F_{xx}} \right)^{3/2}, \end{aligned} \quad (j = 1, 2) \quad (3.9c - e)$$

The system (3.8,9) together with (3.6) then specifies a free-boundary problem for  $\xi_1(x)$ ,  $\xi_2(x)$  and the functions  $A(x)$ ,  $B(x)$  together with the skin friction  $\bar{u}_\xi(x, 0)$ . The solution of this system is a full numerical problem (we note that the similarity reduction available to Hall and Lakin is not applicable here). Before proceeding with a discussion of the marching procedure used to solve this system we note that, in order to solve (3.8,9), it is necessary to specify an ‘initial state’ for the mean velocity field from which the solution can be obtained in a systematic way by marching downstream. This requires finding an asymptotic form for the solution close to the position  $x^*$  at which the original boundary layer becomes unstable.

We assume that the unperturbed boundary layer flow first becomes unstable at the position  $x = x^*$  and that the instability originates in a thin layer centred on  $\xi = \xi^*$ . We define  $X = x - x^* \ll 1$  together with the similarity variable  $\eta = (\xi - \xi^*)/X^{1/2}$  and expand  $\xi_1$  and  $\xi_2$  in the form

$$\begin{aligned} \xi_1 &= \xi^* - X^{1/2} \tilde{\xi} + O(X), \\ \xi_2 &= \xi^* + X^{1/2} \tilde{\xi} + O(X), \end{aligned} \quad (3.10)$$



(these expansions follow from the weakly nonlinear theory of Hall (1982b)). Consider first the core region; following Hall and Lakin (1988) we express  $(\bar{u}_0, \bar{v}_0)$  in the form

$$\bar{u}_0 = \bar{u} + X^{3/2}U_M(\eta) + \dots, \quad \bar{v}_0 = \bar{v} + XV_M(\eta) + \dots, \quad (3.11a, b)$$

where  $(\bar{u}, \bar{v})$  are the boundary layer velocity components in the absence of the vortex motion. In a small neighbourhood of the neutral location  $(x^*, \xi^*)$  we have the Taylor series expansions for  $(\bar{u}, \bar{v})$  and  $F_{xx}$  given by

$$\begin{aligned} \bar{u} &= u_{00} + X^{1/2}\eta u_{10} + X(\eta^2 u_{20} + u_{01}) + X^{3/2}\eta u_{11} + \dots, \\ \bar{v} &= v_{00} + X^{1/2}\eta v_{10} + X(\eta^2 v_{20} + v_{01}) + X^{3/2}\eta v_{11} + \dots, \\ F_{xx} &= F_0 + XF_1 + \dots. \end{aligned} \quad (3.12a - c)$$

Substituting (3.11a,b) together with (3.12a-c) into (3.3) and equating like powers of  $X$  gives

$$\begin{aligned} 2F_0 u_{00} u_{10} &= 1, \quad u_{10}^2 + 2u_{00} u_{20} = 0, \\ U_M &= \lambda_0 \eta - \lambda_1 \eta^3, \quad V_M = -\frac{\lambda_0 \eta^2}{2} + V_M^0, \end{aligned} \quad (3.13a - c)$$

where

$$\lambda_0 = -u_{11} - \frac{u_{10}u_{01}}{u_{00}} - \frac{F_1 u_{10}}{F_0}, \quad \lambda_1 = \frac{u_{10}u_{20}}{u_{00}},$$

and  $V_M^0$  is a constant of integration (required to ensure continuity of  $V_M$  across the shear layers). We note that (3.13a) is automatically satisfied as  $x^*$  is the neutral position, while (3.13b) is satisfied since  $\xi^*$  is chosen so that (2.13) holds.

Following Hall and Lakin (1988) we can show that the vortex amplitude  $V_{01}$  vanishes at  $\eta = \eta_{\pm}$  where

$$|\eta_{\pm}|^2 = \frac{C_0 + 2\lambda_0}{6\lambda_1 + u_{00}\lambda_0}, \quad (3.14)$$

with  $C_0$  an unknown constant to be determined. It then follows from (3.10) and (3.14) that

$$\eta_- = -\tilde{\xi}, \quad \eta_+ = \tilde{\xi}.$$

To determine  $\eta_{\pm}$  it remains to determine the constant  $C_0$ .

Outside the region of vortex activity we now write

$$u = \bar{u} + X^{3/2}U_M(\eta) + \dots, \quad v = \bar{v} + XV_M(\eta) + \dots.$$

We note that the functions  $U_M$  and  $V_M$  are, respectively odd and even functions of  $\eta$  and so we will restrict our attention to the region  $\eta \geq \eta_+$ . Here we find that  $U_M, V_M$  satisfy<sup>†</sup>

$$\begin{aligned} \frac{d^2 U_M}{d\eta^2} + \frac{u_{00}}{2} \left( \eta \frac{d}{d\eta} - 3 \right) U_M &= 0, \\ \frac{dV_M}{d\eta} + \frac{3}{2} U_M - \frac{\eta}{2} \frac{dU_M}{d\eta} &= 0, \end{aligned} \quad (3.15)$$

which are to be solved subject to the matching conditions

$$U_M = \lambda_0 \eta_+ - \lambda_1 \eta_+^3, \quad U'_M = \lambda_0 - 3\lambda_1 \eta_+^2, \quad (3.16a)$$

and

$$U_M \rightarrow 0 \quad \eta \rightarrow \infty. \quad (3.16b)$$

The appropriate solutions are found to be<sup>‡</sup>

$$\begin{aligned} U_M &= \frac{(\lambda_0 \eta_+ - \lambda_1 \eta_+^3) \exp \{ -u_{00}(\eta^2 - \eta_+^2)/8 \} U(7/2, \sqrt{\frac{u_{00}}{2}} \eta)}{U(7/2, \sqrt{\frac{u_{00}}{2}} \eta_+)}, \\ V_M &= \frac{\eta}{2} U_M - 2 \int_{\infty}^{\eta} U_M d\eta, \end{aligned} \quad (3.18a, b)$$

where  $U(a, x)$  is the parabolic cylinder function. From (3.17a) and (3.18a) we then have the eigen-relation<sup>†</sup> for  $\eta_+$

$$-\sqrt{\frac{u_{00}}{2}} \left\{ \frac{4U(9/2, \sqrt{\frac{u_{00}}{2}} \eta_+)}{U(7/2, \sqrt{\frac{u_{00}}{2}} \eta_+)} + \sqrt{\frac{u_{00}}{2}} \eta_+ \right\} = \frac{\lambda_0 - 3\lambda_1 \eta_+^2}{\lambda_0 \eta_+ - \lambda_1 \eta_+^3}, \quad (3.19)$$

which then determines the constant  $C_0$  in (3.14).

With the above asymptotic solution, valid near  $x^*$ , we can construct the composite velocity field together with initial values for the function  $A(x), B(x), \xi_1(x)$  and  $\xi_2(x)$  which will then be used to start the marching procedure to be described in the next section.

<sup>†</sup>Note the typographical error in the corresponding expressions of Hall and Lakin (1988).

<sup>‡</sup>ibid.

#### 4. Numerical solution of the free boundary problem

The numerical scheme used to solve the free boundary problem (3.8-9) is based on that used by Hall and Lakin (1988); for this reason we only give the relevant details here. We assume that a solution is known for  $x^* \leq x \leq \tilde{x}$  (which in practice must be constructed from the asymptotic solution for the initial development of the nonlinear vortex state presented above). The scheme then advances this ‘initial’ solution downstream to  $\tilde{x} + \epsilon$  for some sufficiently small value of  $\epsilon$ .

In the course of our calculations it was found appropriate to solve a modified system of the form outlined in §2. Thus we define

$$\bar{u} = Fu, \quad \bar{v} = v - F_x \xi.$$

The numerical scheme then proceeds by defining a new variable  $\psi$  by

$$\psi = \frac{\xi}{\xi_j}; \quad j = 1, \quad 0 \leq \xi \leq \xi_1, \quad j = 2, \quad \xi_2 \leq \xi \leq \infty.$$

The modified form of the system (3.12) is now written as

$$\begin{aligned} Fu \frac{\partial u}{\partial x} - \frac{1}{\xi_j^2} \frac{\partial^2 u}{\partial \psi^2} &= F_x(1 - u^2) - \frac{(v - F_x \xi_j \phi)}{\xi_j} \frac{\partial u}{\partial \psi} + \frac{F \xi_j'}{\xi_j} \psi u \frac{\partial u}{\partial \psi}, \\ \frac{\partial v}{\partial \psi} &= \xi_j \left( F_x(1 - u) - F \frac{\partial u}{\partial x} + \frac{\psi F \xi_j'}{\xi_j} \frac{\partial u}{\partial \psi} \right), \end{aligned} \quad (4.1a, b)$$

where  $j = 1$  for  $0 \leq \psi < 1$  and  $j = 2$  for  $1 < \psi < \infty$ . This system is to be solved on  $(0, \infty)$  in terms of  $\psi$  with boundary conditions

$$u = v = 0, \quad \psi = 0, \quad u \rightarrow 1 \quad \psi \rightarrow \infty, \quad (4.2)$$

together with the ‘discontinuous’ conditions at  $\psi = 1$

$$\begin{aligned} u_{\pm} &= \frac{1}{F} \sqrt{\frac{A + \xi_j}{F_{xx}}}, \quad u_{\eta \pm} = \frac{\xi_j}{2F \sqrt{F_{xx}(A + \xi_j)}}, \\ v_{\pm} &= -B - A_x \sqrt{\frac{A + \xi_j}{F_{xx}}} + \frac{F_{xxx}}{3} \left( \frac{A + \xi_j}{F_{xx}} \right)^{3/2} + F_x \xi_j. \end{aligned} \quad (4.3a, b)$$

Here the  $\pm$  signs corresponds to the limits  $\psi \rightarrow 1^+$ ,  $\psi \rightarrow 1^-$  while the index  $j = 1$  is associated with the ‘-’ sign and  $j = 2$  with the ‘+’ sign. The system is then closed by the condition (3.6).

The solution of (4.1a) is advanced for  $\psi$  in the range  $0 \leq \psi \leq 1$  by using the scheme

$$\begin{aligned} \tilde{F}u_n\tilde{u}_n - \frac{\epsilon}{h^2\xi_j^2}(\tilde{u}_{n+1} - 2\tilde{u}_n + \tilde{u}_{n-1}) &= Fu_n^2 - \epsilon F_x(1 - u_n^2) \\ - \frac{\epsilon}{2\xi_j h} (v_n - F_x\xi_j\psi_n - F\psi_n\xi_j'u_n)(u_{n+1} - u_{n-1}), \end{aligned} \quad (4.4)$$

where  $h$  is the vertical grid spacing, a tilde denotes that the quantity is evaluated at  $\tilde{x} + \epsilon$  while an index  $n$  indicates a quantity evaluated at the grid point  $\psi = nh$ . In order to solve the tridiagonal system (4.4) we must make an initial guess for  $\tilde{\xi}_1$  and then set  $\xi'_1 = (\tilde{\xi}_1 - \xi_1)/\epsilon$ . In solving (4.4) we ensure that the boundary condition on  $u$  at  $\psi = 0$  and (4.3b), with  $j = 1$ , are satisfied. The solution of (4.4) then yields  $\tilde{u}_n$ . We discretize the modified continuity equation (4.1b) as

$$\frac{\tilde{v}_{n+1} - \tilde{v}_{n-1}}{2h} = \xi_j \left\{ \tilde{F}_x(1 - \tilde{u}_n) + \frac{\tilde{F}(u_n - \tilde{u}_n)}{\epsilon} + \frac{\tilde{F}\psi_n\xi'_j}{\xi_j} \frac{(\tilde{u}_{n+1} - \tilde{u}_{n-1})}{2h} \right\}, \quad (4.5)$$

which, given  $\tilde{u}_n$  from (4.4), allows us to determine  $\tilde{v}_n$  at the position  $\tilde{x} + \epsilon$  for  $0 \leq \psi \leq 1$ . As the equation for  $\tilde{v}$  is first order in  $\psi$  only the boundary condition on  $\tilde{v}$  at  $\psi = 0$  can be satisfied. Thus, although the solution of (4.1), for  $\psi$  in the range  $0 \leq \psi \leq 1$ , has been obtained at  $\tilde{x} + \epsilon$  the additional boundary conditions (4.3a,c), with  $j = 1$ , have not been satisfied. These conditions can now be used to obtain a value for  $\tilde{A}$  and an improved value for  $\tilde{\xi}_1$  by writing them in the form

$$\begin{aligned} \tilde{\xi}_1 &= \tilde{F}^2 \tilde{F}_{xx} \tilde{u}_-^2 - \tilde{A}, \\ \tilde{v}_- + \tilde{B} &= -\frac{(\tilde{A} - A)}{\epsilon} \sqrt{\frac{A + \tilde{\xi}_1}{\tilde{F}_{xx}}} + \frac{\tilde{F}_{xxx}}{3} \left( \frac{A + \tilde{\xi}_1}{\tilde{F}_{xx}} \right)^{3/2} + \tilde{F}_x \tilde{\xi}_1, \end{aligned} \quad (4.6a, b)$$

Here  $\tilde{B}$  is the current guess of  $B$  at  $\tilde{x} + \epsilon$ . This scheme can be applied in a similar manner to the region  $\psi \geq 1$ . The equation (4.4) is now solved for  $\tilde{u}$  subject to  $\tilde{u} \rightarrow 1$  as  $\psi \rightarrow \infty$

together with (4.3b) for  $j = 2$ , and equation (4.5) is solved for  $\tilde{v}$  subject to (4.3c) with  $j = 2$ . Writing (4.3a) and (3.6) in the form

$$\begin{aligned}\tilde{\xi}_2 &= \tilde{F}^2 \tilde{F}_{xx} \tilde{u}_+^2 - \tilde{A}, \\ -\tilde{B} \left\{ \sqrt{\frac{\tilde{A} + \tilde{\xi}_1}{\tilde{F}_{xx}}} - \sqrt{\frac{\tilde{A} + \tilde{\xi}_2}{\tilde{F}_{xx}}} \right\} &= \frac{\tilde{F}_{xxx}}{6\tilde{F}_{xx}^2} \left\{ (\tilde{A} + \tilde{\xi}_1)^2 - (\tilde{A} + \tilde{\xi}_2)^2 \right\} \\ &+ \frac{1}{2\sqrt{\tilde{F}_{xx}}} \left\{ \frac{1}{\sqrt{\tilde{A} + \tilde{\xi}_1}} - \frac{1}{\sqrt{\tilde{A} + \tilde{\xi}_2}} \right\} + \tilde{F}\tilde{F}_x(\tilde{\xi}_1 - \tilde{\xi}_2),\end{aligned}\tag{4.6c, d}$$

the first of which then determines a new value for  $\tilde{\xi}_2$  while the second equation yields a value for  $\tilde{B}$ . We then have values of  $u, v, A, B, \xi_1$  and  $\xi_2$  at  $\tilde{x} + \epsilon$ . This procedure is then continued, making use equations (4.6a-d), using Newton iteration on  $\tilde{A}, \tilde{B}, \tilde{\xi}_1$  and  $\tilde{\xi}_2$  until converged values of these quantities are obtained.

For definiteness, our numerical calculations were restricted to the profile (2.11). The first stage in the numerical solution of the free boundary problem (3.8,9) involves determining the neutral position from (2.12,13). This was achieved using the numerical scheme outlined in §2; with the initial conditions  $\bar{u} = \bar{v} = 0$  at  $x = 0$  the solution was marched downstream, at each stream-wise location  $\xi^*$  was determined from (2.13), until the point of neutral stability  $x^*$ , determined from (2.12), was reached. At this point an initial guess for  $\xi_1(\tilde{x})$ ,  $\xi_2(\tilde{x})$ ,  $A(\tilde{x})$  and  $B(\tilde{x})$ , for  $\tilde{x} = x^* + 0.001$ , was obtained using the procedure outlined at the end of §3. The initial condition required for the velocity field at  $\tilde{x}$ , namely  $\bar{u}(\tilde{x}, \xi)$  and  $\bar{v}(\tilde{x}, \xi)$ , were obtained using the scheme (2.10) and these values were then interpolated, using cubic spline interpolation in order to ensure continuous second derivatives, to yield  $\bar{u}$  and  $\bar{v}$  on a uniform grid in  $\psi$  in the intervals  $[0, 1)$  and  $(1, \psi_\infty)$ . A similar interpolation procedure was used on the vortex induced mean flow corrections  $U_m, V_m$  thus allowing the composite velocity field to be computed. The composite velocity field so obtained was then used as the initial conditions for the marching scheme described above. The solution was marched downstream until such point as the limit of the region of vortex activity tended to zero (ie at the point  $\tilde{x}_F$  where  $\xi_1(\tilde{x}_F) = \xi_2(\tilde{x}_F)$ , see figure 4). At this point the velocity

field was then used as the initial condition for the scheme (2.10) which was then used to march downstream until such point at which reversed flow is first encountered.

## 5. Conclusions

Before proceeding with a discussion of our results we make some comments concerning the stability characteristics of our numerical scheme. Initially the normal step-size  $h$  was fixed at  $h = 0.01$  with  $\psi_\infty = 16$  thus yielding 100 and 1500 normal grid points in the intervals  $0 \leq \psi \leq 1$  and  $1 \leq \psi \leq \psi_\infty$ , respectively. With  $h$  fixed, various calculations were performed with different values of the streamwise step-length  $\epsilon$ ; from  $\epsilon = 0.001$  to  $\epsilon = 0.00025$ . For the range of  $\epsilon$  considered the position  $x_F$ , at which the vortex region vanishes, was found to vary by, typically, 0.9% over the value obtained for the smallest value of  $\epsilon$  considered. Similarly, the normal position  $\xi_F$  was found to vary by 0.7% over the range of  $\epsilon$  considered. We deem our numerical scheme to be stable and for  $\epsilon = 0.00025$  suitable accuracy has been attained. All of the subsequent results were obtained for the choice of parameters  $h = 0.01$ ,  $\epsilon = 0.00025$  and  $\psi_\infty = 16$ .

Consider figure 5, in which we present a plot of the positions  $\xi_1(x)$  and  $\xi_2(x)$  bounding the region of vortex activity for the constriction amplitudes  $\alpha = 1.0304$  and  $\alpha = 1.0923$ . As anticipated from the results of Hall and Lakin (1988) we see that the region of vortex activity initially thickens until such a point at which the maximum normal extent of this region is achieved. As  $x$  is further increased, the region of vortex activity constricts until the point  $x_F$  at which  $\xi_1(x_F) = \xi_2(x_F)$ . For  $x > x_F$  the flow can no longer support the large amplitude vortex state described above.

The effect of increasing the amplitude  $\alpha$  of the constriction is evident from a comparison of figures 5a,b. The minimum values of  $\xi_1$  in figures 5a,b are  $2.19 \times 10^{-2}$  and  $1.08 \times 10^{-2}$  respectively whereas the maximum values of  $\xi_2$  are 1.384 and 1.752 respectively. Thus the effect of increasing  $\alpha$  is twofold; firstly the streamwise extent of the vortex region is increased with increasing  $\alpha$  (a consequence of the change in the neutral positions as represented in figure 3) and secondly, the normal extent of the vortex region increases with increasing  $\alpha$ , through both the increase in the maximum value of  $\xi_2$  and a decrease in the minimum value of  $\xi_1$ .

Turning now to the question as to what effect the nonlinear vortex state, present in the interval  $(x^*, X_F)$ , has upon the onset of separation we present, in figures 6a-b, plots of the skin friction in the presence of the vortex (broken line) and, for comparison, the corresponding skin friction for the unperturbed boundary layer (solid line), for the representative cases  $\alpha = 1.0304$  and  $\alpha = 1.0923$ . Here the skin friction is defined according to

$$\tau = F(x) \frac{\partial u(x, 0)}{\partial \xi} = \frac{F(x)}{\xi_1(x)} \frac{\partial u(x, 0)}{\partial \psi}, \quad (5.1)$$

where  $\psi$  is the modified independent variable and  $u$  is the modified dependent variable used in the numerical scheme described above. From a computational standpoint we replace (5.1) by

$$\tau = \frac{F(x)}{\xi_1(x)} \frac{\partial u(x, 0)}{\partial \psi} = \frac{F(x)}{2h\xi_1(x)} \{4u(x, h) - u(x, 2h)\} + O(h^2). \quad (5.2)$$

The effect on the boundary layer due to the vortex is readily seen through the dramatic increase in the skin friction (over that for the unperturbed state). In the presence of the vortex the skin friction now develops two maxima, the first being due to the vortex, whereas the second is seen to coincide with that for the unperturbed state. The ratio of these maxima is seen to increase with increasing  $\alpha$ ; for the case of figure 6a this is found to be 73% while for figure 6b is 93%. However, in both figures 6a,b we see that the vortex induced skin friction subsequently decreases to below the value found for the unperturbed state followed by an increase, due to the presence of the favourable pressure gradient, until such point as the second maxima is attained (at which point the difference between the vortex induced state and the unperturbed state is barely discernible). The skin friction now decreases until the point of separation is encountered, the location of which is identical for both the unperturbed state and the case in which the flow supports a nonlinear vortex.

We are then given to conclude that the presence of a nonlinear vortex, in a region of concave curvature upstream of the position of maximum channel constriction, has no discernible effect upon the position at which separation is encountered. This result is somewhat surprising in light of the comments of Hall and Lakin (1988) (pp 443) that the decay of the vortices does not allow the boundary layer to return to its undisturbed state. In the present case it would appear, by the position at which separation is reached, that

the boundary layer does, in fact, return to its unperturbed state. In figure 7 we present a plot of the streamwise velocity, for the case of the unperturbed boundary layer and the case when the vortices are present, at the position  $x_F$ .

In the analysis presented we have ignored two other possible mechanisms for linear instability and subsequent nonlinear growth, namely Tollmien–Schlichting waves and inviscid Rayleigh modes (see Tutty and Cowley (1986) for a discussion of the Rayleigh instability in interactive boundary layer flows). The nonlinear development of these modes, together with the possible importance of the vortex–wave interaction mechanisms described by Hall and Smith (1988), could have a dramatic effect on the results and conclusions of this work.

Finally, for the (somewhat more realistic) case of moderately constricted channels, for which  $h = O(1)$ , the effective Görtler number is now an  $O(1)$  quantity and we would expect, on the basis of the linear theory of Hall (1983) for vortices with an  $O(1)$  wavenumber, that the flow will support linearly unstable Görtler modes. The extension into the nonlinear regime then involves the numerical solution of the full nonlinear governing equations as was carried out by Hall (1988) for the external boundary layer problem. In the present case the governing equations show distinct similarities with those derived by Denier and Hall (1992) to describe the nonlinear evolution of the most unstable Görtler vortex mode the solution of which resulted in a separated flow. In the context of an  $O(1)$  constricted channel, the resolution of this problem would require the numerical solution of the full nonlinear governing equations. This is currently the subject of ongoing research.

### Acknowledgements

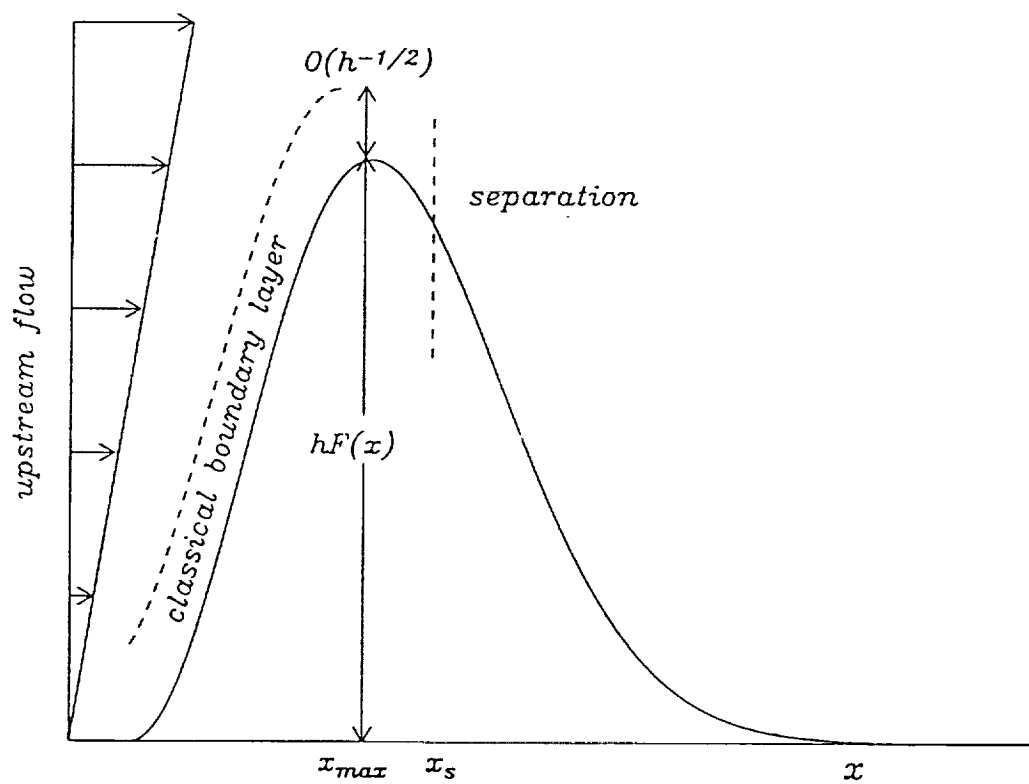
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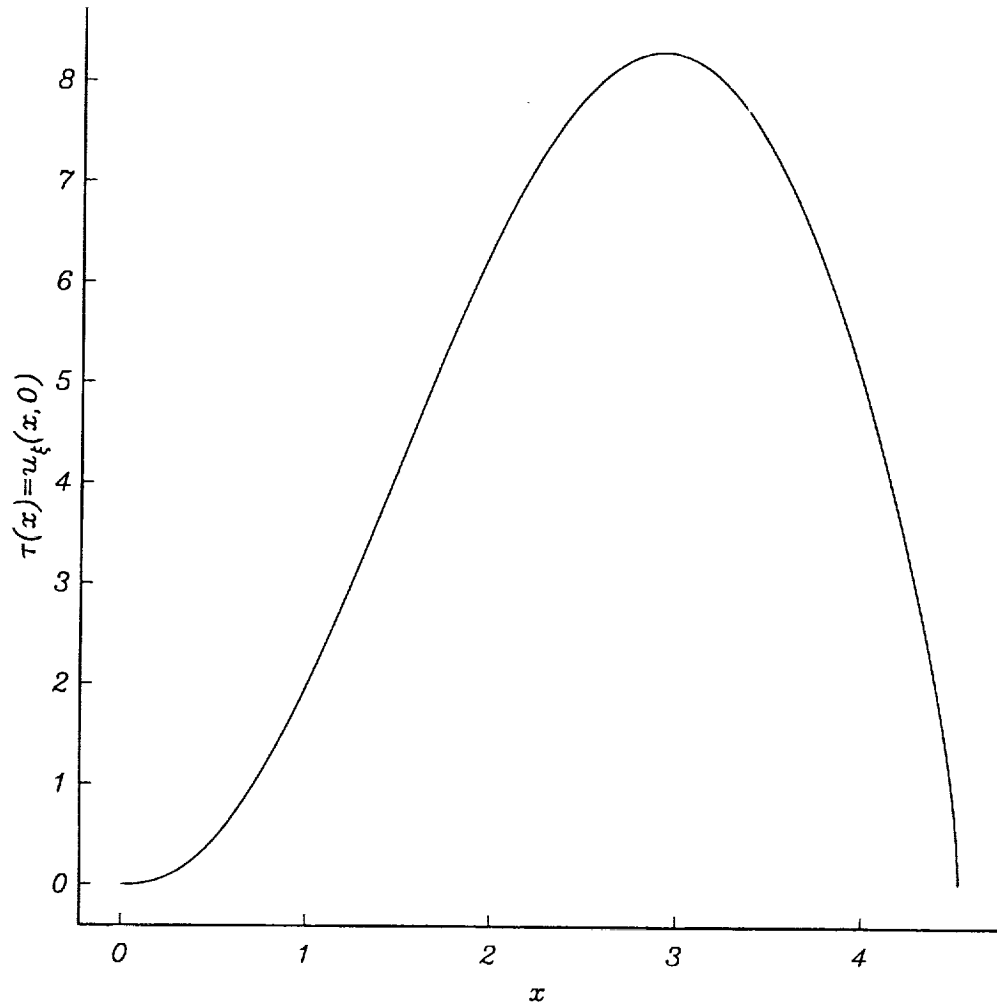
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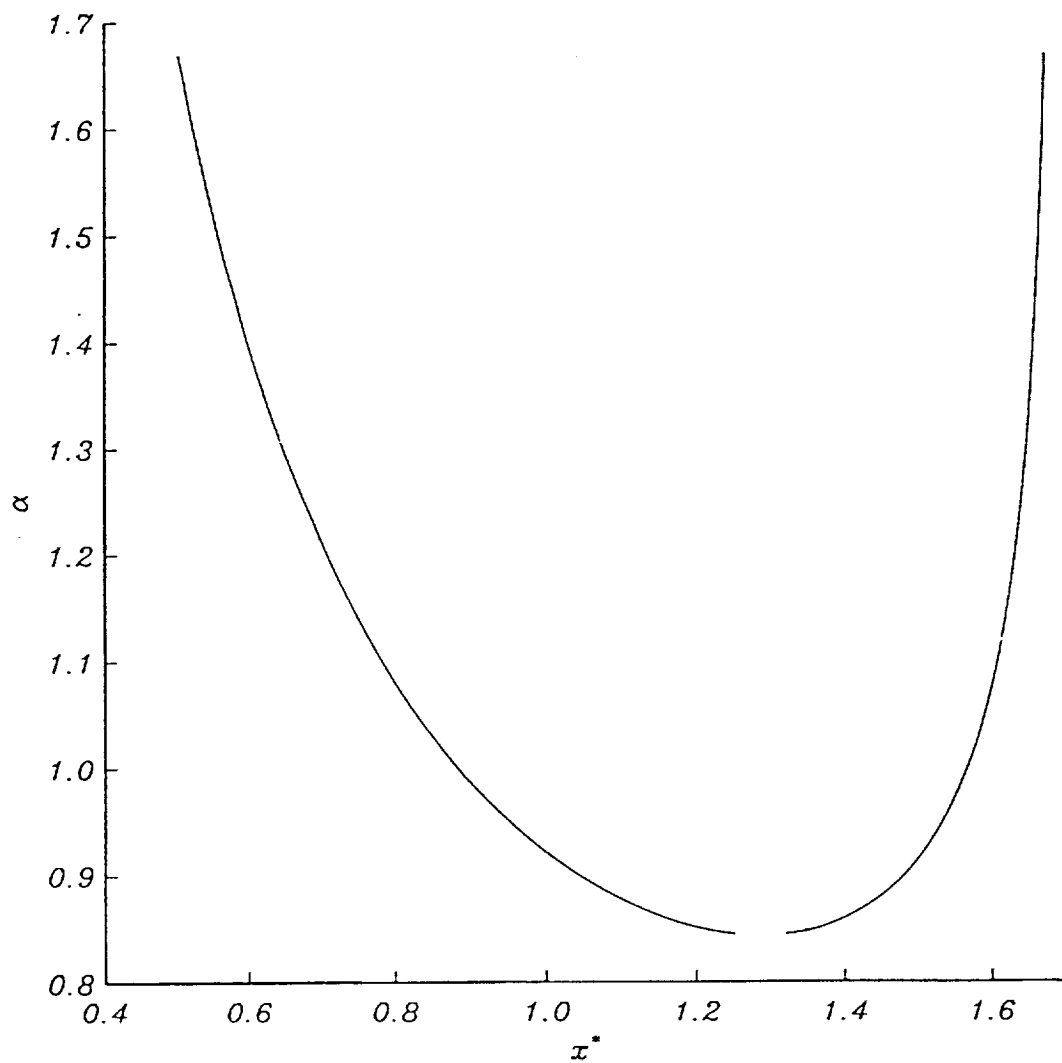
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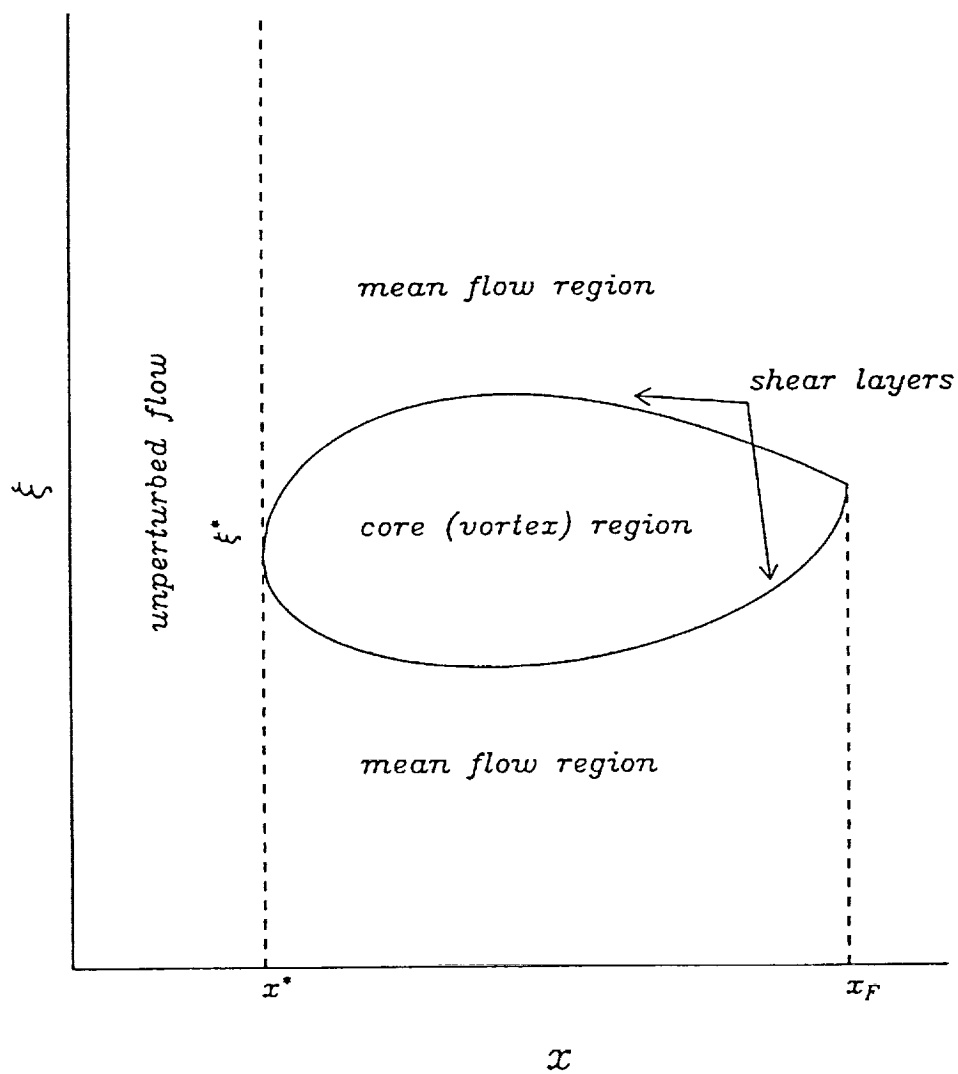
**Figure 1.** A sketch of the flow configuration under consideration.



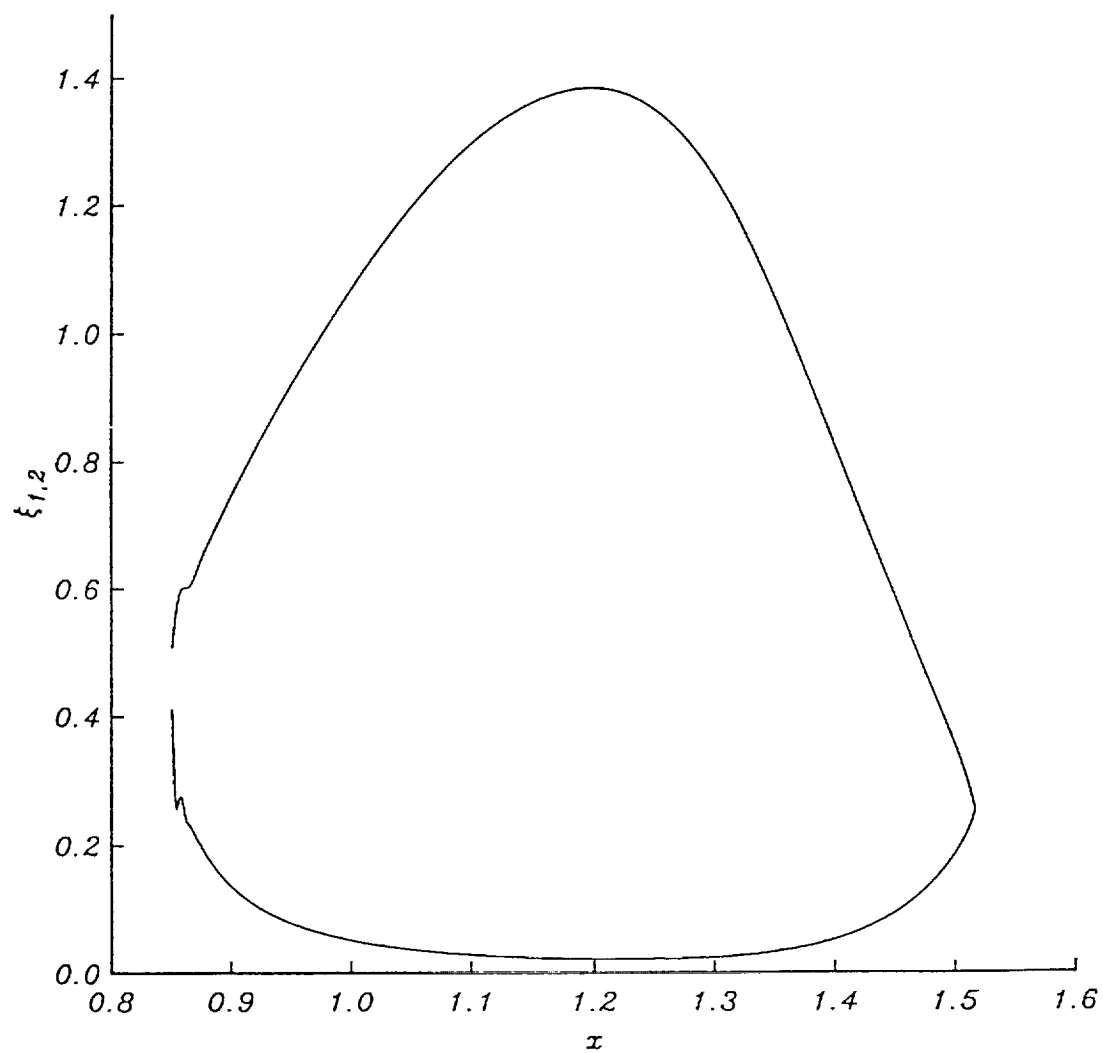
**Figure 2.** Plot of skin friction  $\tau = \bar{u}_\xi(x, 0)$  versus  $x$  for the profile (3.3).



**Figure 3.** The critical position  $x^*$  versus scaled amplitude  $\alpha$ . Note the existence of two critical positions for  $\alpha > \alpha_c$ .



**Figure 4.** A sketch of the various flow regimes.



**Figure 5a.** The region of vortex activity; shown is a plot of  $\xi_1(x)$  and  $\xi_2(x)$  versus  $x$ . (a)  $\alpha = 1.0304$ , (b)  $\alpha = 1.0923$ .

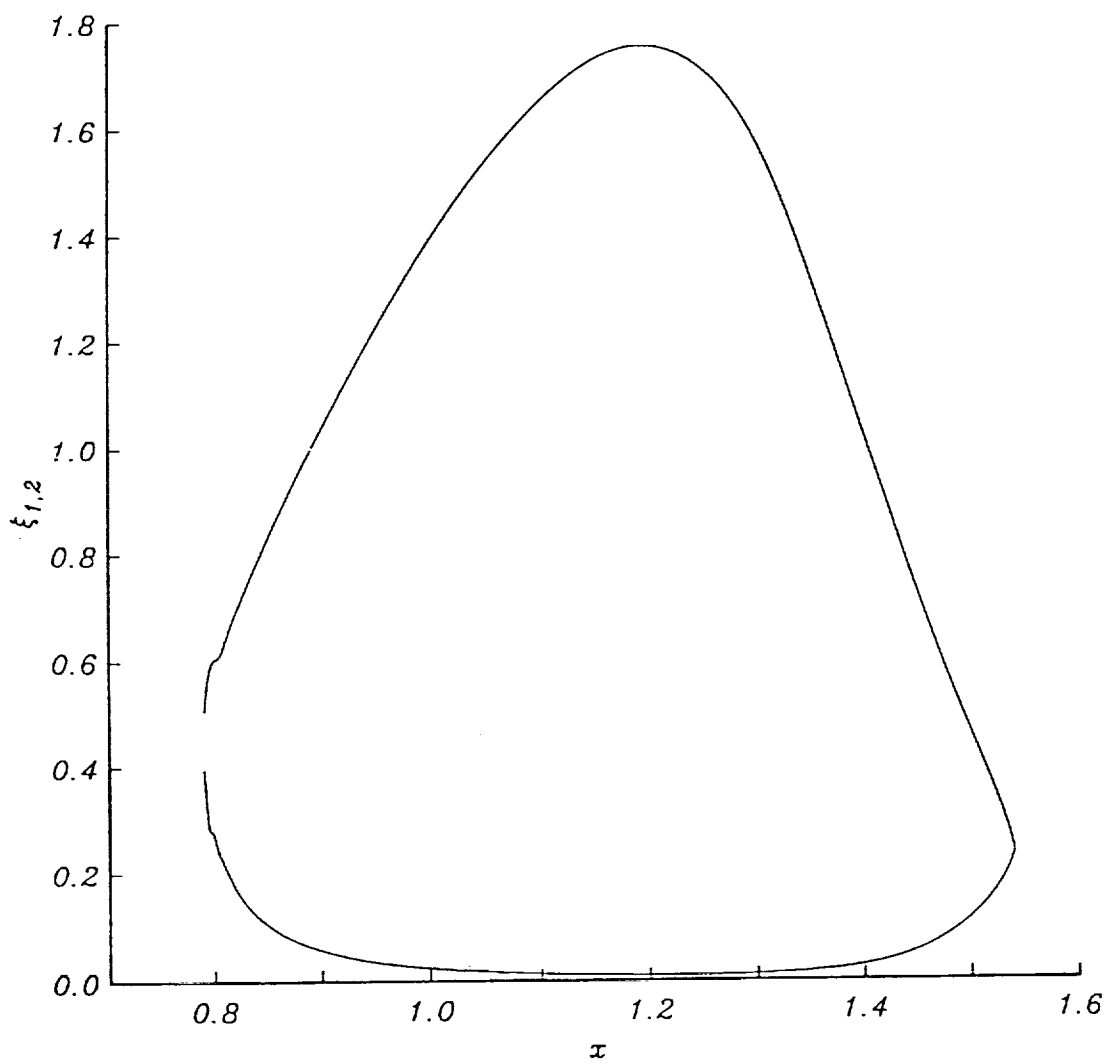
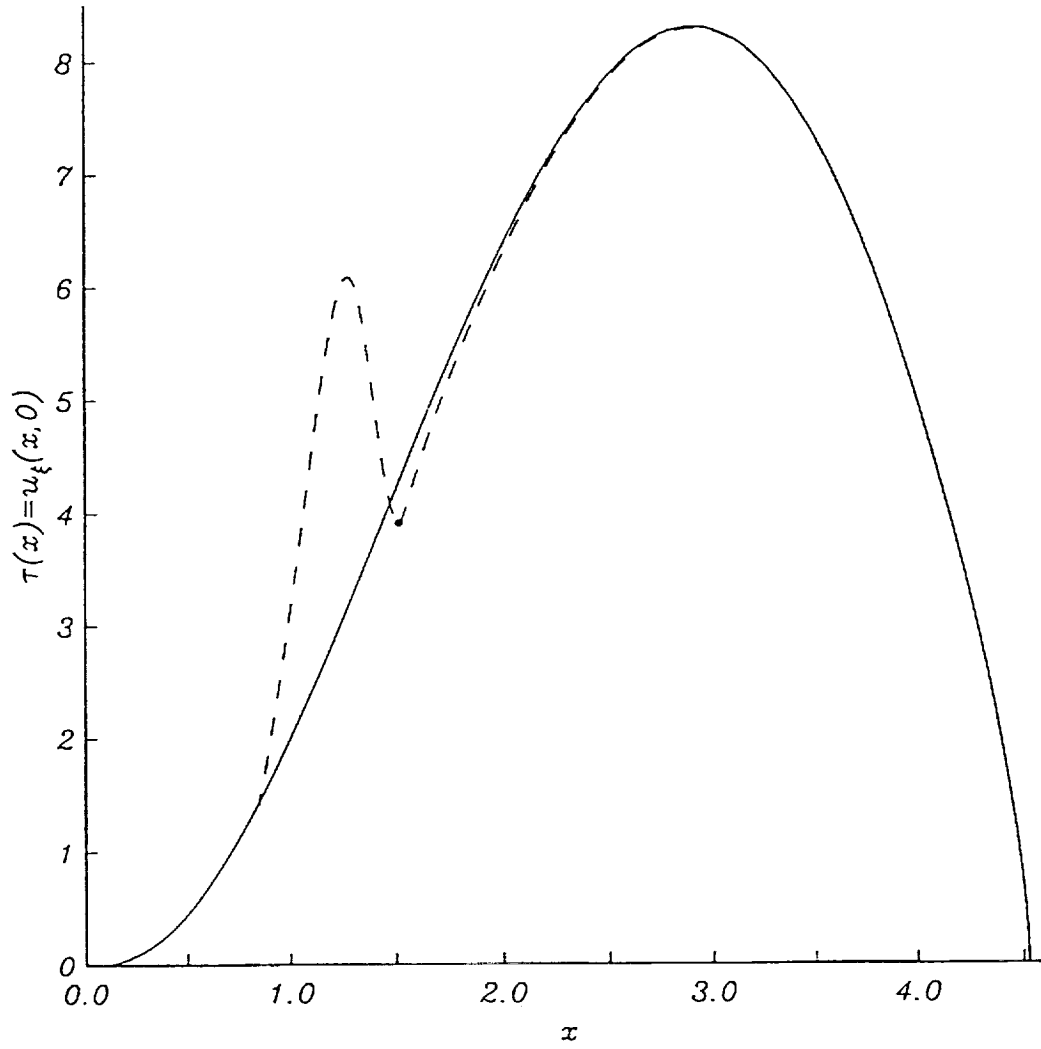


Figure 5b. See figure 5a for legend.





**Figure 6a.** Plot of the skin friction  $\tau$  in the presence of the vortex (broken line). For comparison the skin friction in the absence of the vortex is presented (solid line). The  $\bullet$  represent the position at which the vortex region has decayed to zero (ie  $x_F$ ). (a)  $\alpha = 1.0304$ , (b)  $\alpha = 1.0923$ .

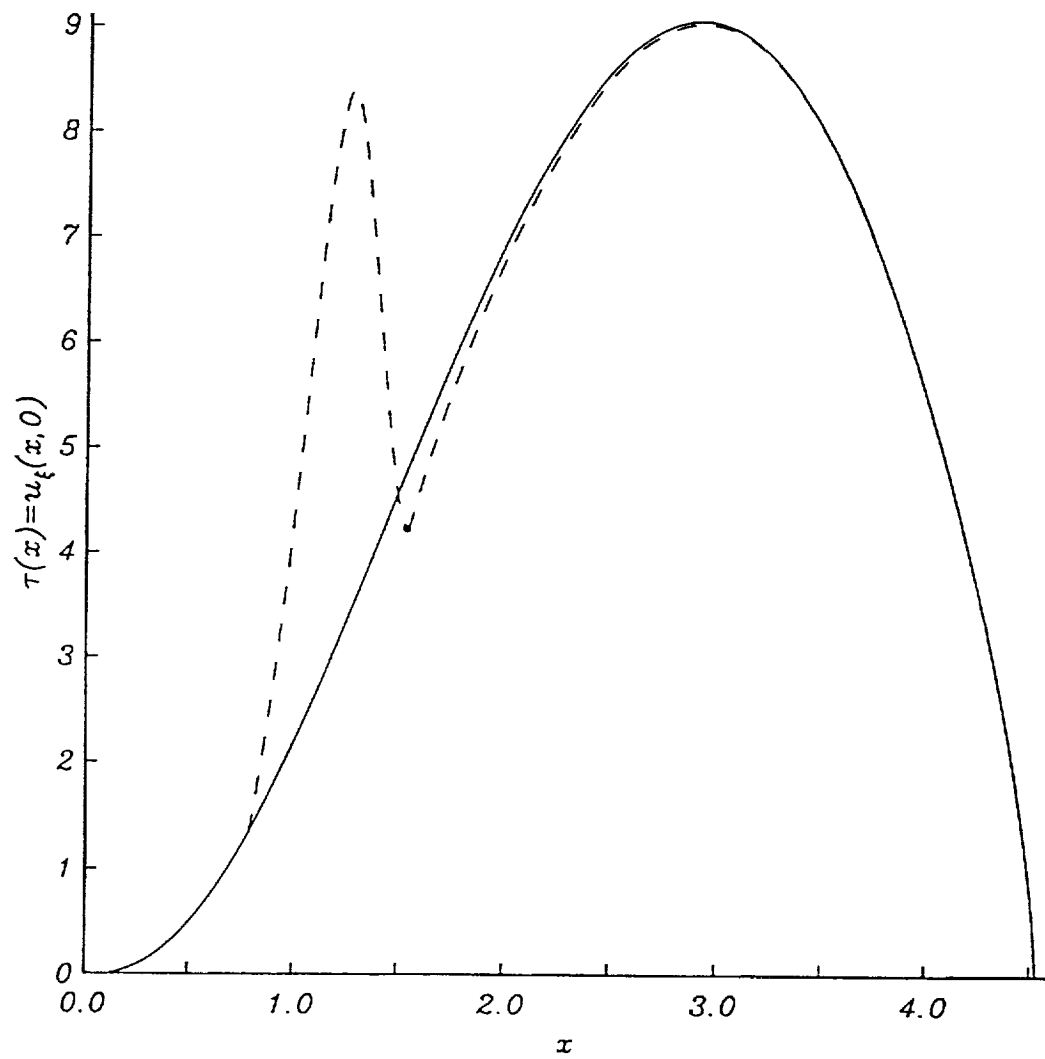
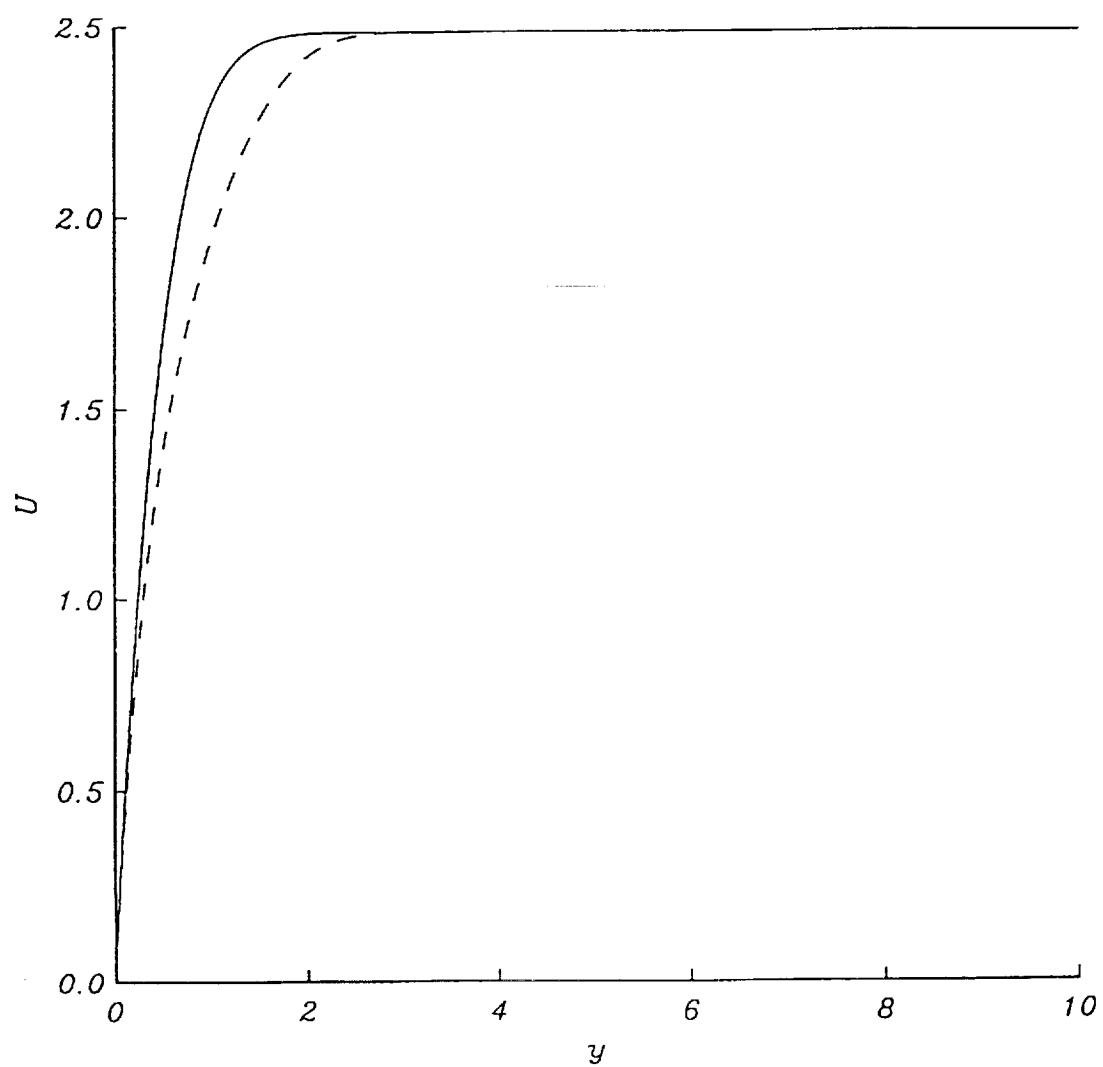


Figure 6b. See figure 6a for legend.



**Figure 7.** Streamwise velocity profiles for  $x = x_F$ . The vortex induced mean flow (broken line) and the unperturbed mean flow (solid line).





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13. ABSTRACT (Maximum 200 words)  The development of fully nonlinear Görtler vortices in the high Reynolds number flow in a symmetrically constricted channel is investigated. Attention is restricted to the case of 'strongly' constricted channels considered by Smith and Daniels (1981) for which the scaled constriction height is asymptotically large. Such flows are known to develop a Goldstein singularity and subsequently become separated at some downstream station past the point of maximum channel constriction. It is shown that these flows can support fully nonlinear Görtler vortices, of the form elucidated by Hall and Lakin (1988), for constrictions which have an appreciable region of local concave curvature upstream of the position at which separation occurs. The effect on the onset of separation due to the nonlinear Görtler modes will be discussed. A brief discussion of other possible nonlinear states which may also have a dramatic effect in delaying (or promoting) separation will be given.				
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